Representations of the Symmetric Group, the General Linear Group, and Related Algebras

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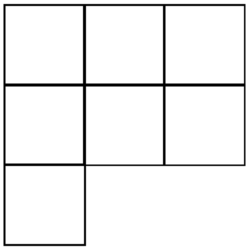
1 Introduction

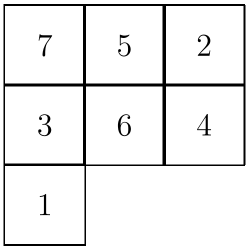
Representation theory studies the properties of groups and rings through their actions on vector spaces. A vector space V with an action is called a representation. If a group G is finite and V is over a field F of characteristic zero, it is known that V is a direct sum of "elementary building blocks" called irreducible representations. If in addition F is algebraically closed, then the number of conjugacy classes in G is equal to the number of distinct irreducible representations, but in general there is no bijection between the two. The basic objects in combinatorial representation theory are the symmetric group $\mathcal{S}_k$ and the general linear group GL_n. The goal of this project will be to generalize the combinatorics used to classify representations of $\mathcal{S}_k$ and GL_n to the study of related algebras. Also, it will be interesting to investigate algebras over fields of finite characteristic. The combinatorics developed for $\mathcal{S}_k$ and GL_n will provide the basic tools for calculating representations. To develop an intuitive understanding of what to expect from each representation I will consider a connection with geometry. This familiarity with representations will be important for any further work in the field.

2 Project Description

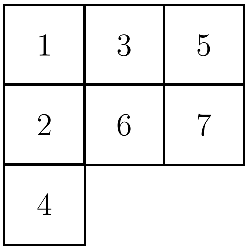
The beginning of the summer will be devoted to learning the basic theory and combinatorial tools used in representations of $\mathcal{S}_k$ and GL_n. In the case of the symmetric group $\mathcal{S}_k$ there exists a bijection between the conjugacy classes and irreducible representations. Both sides of the bijection are given by partitions λ of the integer k. In particular, for every partition λ there is a way to construct an irreducible representation V^λ using combinatorics of the Young tableaux associated with λ. A Young diagram corresponding to a partition $[u_1, ..., u_n]$ is a set of boxes with u_p slots in the p^{th} row. A Young tableau is obtained from the diagram by filling the slots with integers 1-k. For example, the following figure illustrates the diagram and two sample tableau associated to the partition [3,3,1] of k=7.



$[\lambda] - diagram$



general tableau



standard tableau
(numbers increase down each column and left to right on each row)

From this characterization of the irreducibles one can compute the dimension(“hook lengt”) and a basis (“standard tableaux”) in terms of the Young tableau. Finally, given a representation V_1 of $\mathcal{S}_{k-r}$ and V_2 of $\mathcal{S}_r$ we want to consider $V_1 \otimes V_2$ as a representation of $\mathcal{S}_{k-r} \times \mathcal{S}_r$. Letting $V_1 \star V_2$ denote the representation of $\mathcal{S}_k$ induced by the embedding of $\mathcal{S}_{k-r} \times \mathcal{S}_r \hookrightarrow \mathcal{S}_k$, the Littlewood-Richardson rule gives a method of computing $V_1 \star V_2$ in terms of irreducible representations of $\mathcal{S}_k$. The above theory is discussed in Crawley-Boevey [**?**], James [**?**], and Fulton [**?**] and I will begin by reading these references.

The next step will be to study rational representations of GL_n. For this part I refer to Crawley-Boevey[1], Fulton and Harris [**?**], and Barcelo and Ram [**?**] The Weyl Character formula will serve a similar purpose for irreducible representations of $GL(n, \mathbb{C})$ as did the hook-length formula for representations of $\mathcal{S}_k$. It is possible to construct bases in representations of $GL(n, \mathbb{C})$ that have nice combinatorial properties, and are well-behaved with respect to taking tensor products. These base provide further insight in to the Schur-Weyl duality and the Littlewood-Richardson rule. The connection with representations of $\mathcal{S}_k$ is given by the Schur-Weyl duality:

1. *The action of* $\mathcal{S}_k$ *on* $V^{\otimes k}$ *generates* $End_{GL(n,\mathbb{C})} V^{\otimes k}$
2. *The action of* $GL(n, \mathbb{C})$ *on* $V^{\otimes k}$ *generates* $End_{\mathcal{S}_k}(V^{\otimes k})$

This provides a correspondence between the rational representations of $GL(n, \mathbb{C})$ obtained from $V^{\otimes k}$ and representations of $\mathcal{S}_k$. The combinatorial techniques developed above will enable me to study representations of related algebras. In particular, I will consider deformations of $\mathbb{C}\mathcal{S}_k$ and the infinite-dimensional analogue of the Lie Algebra $gl(n, \mathbb{C})$ of $GL(n, \mathbb{C})$. In addition, I will consider how representations arise from the geometry of topological spaces with a group acting on them.

3 Expected Outcome

The beginning of the summer will be devoted to learning the combinatorial tools necessary to study representations of $\mathbb{C}\mathcal{S}_k$ and $GL(n, \mathbb{C})$. This knowledge will provide the basis for entering active areas of research in representation theory. In particular, I will be able to start experimenting with various representations and hopefully investigate some novel ones. For example, I will study the Lie Algebra gl_∞ and, possibly, some affine Lie algebras. Finally, I want to relate this theory to the geometry of topological spaces with a group acting on them (“Borel-Weil theory” and “geometry of Quiver Varieties”). The latter will be an interesting extension the study of algebraic structures to geometric objects.

References

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[2] Crawley-Boevey, William. Lectures on Representation Theory and Invariant Theory. 1990, available on author's website: http://www.amsta.leeds.ac.uk/Pure/staff/crawley_b/.

[3] Fulton, William, and Joe Harris. Representation Theory, A First Course. Graduate Texts in Mathematics, Readings in Mathematics. New York: Springer-Verlag, 1991.

[4] Fulton, William. Young Tableaux: With applications to representation theory and geometry. New York: Cambridge University Press, 1997.

[5] James, G.D. The Representations Theory of the Symmetric Groups. Lecture Notes in mathematics. Ed. A. Dold and B. Eckman. New York: Springer-Verlag, 1978.