Part II

Chapter 1

5 *Given a Lie group G, define its identity component G_0 to be the connected component containing the identity element. Show that the identity component of any Lie group is a subgroup, and a Lie group in its own right.*

Suppose we have a path from the identity to $g \in G_0$. Now map this path to a new path by multiplying each element by $h \in G_0$. This path starts at h and since the mapping is continuous, must remain in G_0. (Otherwise, smoothly mapping the group manifold to $\mathbb{R}^n$ would show a discontinuity at some point.) Thus $hg \in G_0$ for all $h, g \in G_0$.
There's a certain tension between having a smooth manifold with disconnected pieces — given a map from the the manifold to itself, one must take care that it does not have a discontinuous action, mapping some points in one component to another component. When this map is an element of the group, this requirement makes G_0 into a subgroup.

6 *Show that every element of O(3) is either a rotation about some axis or a rotation about some axis followed by a reflection through some plane. Show that the former class of elements are all in the identity component of O(3), while the latter are not. Conlude that the identity component of O(3) is SO(3).*

Rotations preserve the inner product between vectors, so with any rotation R we can transform the standard orthonormal basis $\{e_1, e_2, e_3\}$ to another. But Re_1 is just the first column of R, so this fact means that $R^T R = \mathbb{1}$. Conversely, the condition $Q^T Q = \mathbb{1}$ means that Q can be thought of as a set of orthonormal basis elements, and the action of Q on the standard basis will produce this new basis. This new basis need not be a rotated version of the standard basis, however, as it could also involve reflections. This corresponds to a negative determinant as the orientation of the standard basis is not preserved. Aany such Q can be decomposed into $Q = PR$ by using a reflection operator. Let P' be the reflection through the plane defined by the first two standard basis vectors, i.e. the one flipping the third standard basis vector. Clearly $QP' = R$ for some rotation R, since it now has a unit determinant. Then write $QP' = QP'Q^TQ = PQ$, defining $P = QP'Q^T$, which is symmetric: $P^T = QP'^TQ^T = QP'Q^T = P$. Thus $Q = PR$ for some rotation R and reflection P. All rotations are connected to the identity, since we can imagine continuously varying the rotation amount. Thus the identity-connected component is SO(3). The reflections could not be the SO(3) since they do not form a subgroup; the product of two reflections is a rotation.

8 *Show that if $\rho : G \to H$ is a homomorphism of groups, then $\rho(1) = 1$ and $\rho(g^{-1}) = \rho(g)^{-1}$.*

First, the identity and inverses in a group are unique. Suppose there were two identity elements, e and 1. Then $ge = eg = g = g1 = 1g$ for all g and in particular $e = e1 = 1$, using the identity properties of first 1 and then e. The uniqueness of the identity implies the uniqueness of inverses, for suppose g had two inverses, h and k. Then $gh = gk = hg = hk = 1$ and thus $hgh = h = hgk = k$. Now if ρ is to be a homomorphism, then $\rho(g) = \rho(g1) = \rho(g)\rho(1)$ for all g, whence $\rho(1)$ is the identity. Similarly, $\rho(1) = \rho(gg^{-1}) = \rho(g)\rho(g^{-1})$ for all g. So $\rho(g^{-1})$ must be the inverse of $\rho(g)$, i.e. $\rho(g)^{-1}$.

9 *A 1×1 matrix is just a number, so show that U(1)= $\{e^{i\theta} : \theta \in \mathbb{R}\}$. In physics, an element of U(1) is a called a phase. Show that U(1) is isomorphic to SO(2), with an isomorphism being given by*

$$\rho(e^{i\theta}) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

First we check if the map is a homomorphism, and then if it is bijective. For the homomorphism we need to show that $\rho(e^{i\theta})\rho(e^{i\phi}) = \rho(e^{i(\theta+\phi)})$, which follows since

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} = \begin{pmatrix} \cos(\theta+\phi) & \sin(\theta+\phi) \\ -\sin(\theta+\phi) & \cos(\theta+\phi) \end{pmatrix}.$$

The map is onto, since every element in SO(2) is of the given form, for some θ. It is one-to-one since unless $\theta = \phi$, $\rho(e^{i\theta}) \neq \rho(e^{i\phi})$.

12 *Show that for any bilinear function $f : V \times V' \to W$ from vector spaces V and V' to W there exists a unique linear function $F : V \otimes V' \to W$ such that $f(v, v') = F(v \otimes v')$.*

Here we're not trying to prove abstractly that the tensor product has this property; this property is the definition of tensor product. We want to show this property holds for the tensor product given. For the function f to be bilinear, it must obey $f(v, v') = f(v^i e_i, v'^j e'_j) = v^i v'^j f(e_i, e'_j)$. But $v \otimes v' = v^i v'^j e_i \otimes e'_j$, so we can just define F by the action on the basis: $F(e_i \otimes e'_j) = f(e_i, e'_j)$. This set is a linearly independent basis, since the parts are linearly independent, and so the function defined this way is unique.

18 *Show that any 2×2 matrix may be uniquely expressed as a linear combination of Pauli matrixes $\sigma_0, \ldots, \sigma_3$ with complex coefficients, and that the matrix is hermitian iff these coefficients are real. Show tha tthe matrix is traceless iff the coefficient of σ_0 vanishes.*

A general 2×2 matrix A has components a_{jk}, i.e. in the "standard basis" of matrices. By direct calculation we can also represent A as $\sum_k c_k \sigma_k$ for $2c_0 = a_{00} + a_{11}$, $2c_3 = a_{00} - a_{11}$, $2c_1 = a_{01} + a_{10}$, and $2c_2 = i(a_{01} - a_{10})$. Hermiticity requires that the c_j are real, since the σ_j are, which can also be seen from direct calculation. c_0 is half the trace, so the matrix is traceless exactly when $c_0 = 0$.

20 *Show that the determinant of the 2×2 matrix $a + bI + cJ + dK$ is $a^2 + b^2 + c^2 + d^2$. Show that if a, b, c, d are real and $a^2 + b^2 + c^2 + d^2 = 1$, this matrix is unitary. Conclude that SU(2) is the unit sphere in $\mathbb{H}$.*

In the usual representation the matrix is just $\begin{pmatrix} a - id & -ib - c \\ -ib + c & a + id \end{pmatrix}$, so the determinant follows. Direct calculation reveals $MM^\dagger = \det(M)\mathbb{1}$. Regarding the matrices I, J, K as the three quaternionic units, which we can do since they obey the same algebra as the matrices, we can see that any element of SU(2) can be regarded as a point on the quaternionic unit sphere.

21 *Show that the spin-0 representation of SU(2) is equivalent to the trivial representation in which every element of the group acts on $\mathbb{C}$ as the identity.*

To construct the representations, we start from the homogeneous polynomials $f(x, y)$ on $(x, y) \in \mathbb{C}^2$ of degree $2j$. This is a vector space of dimension $2j + 1$ since $x^{2j}, x^{2j-1}y, \ldots y^{2j}$ forms a basis. Now for any $g \in$SU(2), let $U_j(g)$ be such that $(U_j(g)f)(v) = f(g^{-1}v)$ for $v \in \mathbb{C}^2$. In the case of spin-0, $f(x, y) = c$ for some constant c. Thus $f(g^{-1}v) = c = (U_0(g)f)(v)$ and we must therefore have $U_0(g)f = f$.

22 *Show that the spin-1/2 represenation of SU(2) is equivalent to the fundamental representation in which every element $g \in$SU(2) acts on $\mathbb{C}^2$ by matrix multiplication.*

Here the homogeneous polynomials are simply $f(x,y) = ax + by$, i.e. elements of $\mathbb{C}^2$. Then regarding f as the column vector $\bar{f}$, $f(x,y)$ becomes the inner product $\bar{f}^T \cdot (x,y)$. $f(g^{-1}v)$ is therefore $\bar{f}^T \cdot g^{-1}v = (g\bar{f})^T \cdot v$, so $U_{1/2}(g)f = gf$.

23 *Show that for any representation ρ of a group G on a vector space V there is a dual or contragredient representation ρ^* of G on V^*, given by*

$$(\rho^*(g)f)(v) = f(\rho(g^{-1})v)$$

for all $v \in V, f \in V^*$. Show that all the representations U_j of SU(2) are equivalent to their duals.

Clearly $\rho^*(e) = 1$, so we need to show that $\rho^*(gh) = \rho^*(g)\rho^*(h)$. $(\rho^*(gh)f)(v) = f(\rho((gh)^{-1})v) = f(\rho(h^{-1}g^{-1})v) = f(\rho(h^{-1})\rho(g^{-1})v) = (\rho^*(g)\rho^*(h)f)(v)$.

Chapter 2

55 *Given a manifold M, define charts for TM starting from charts $\varphi_\alpha : U_\alpha \to \mathbb{R}^n$ as follows. Let V_α be the subset of TM given by $V_\alpha = \{v \in TM : \pi(v) \in U_\alpha\}$. Show that every point in TM lies in some set V_α. Define maps $\psi_\alpha : V_\alpha \to \mathbb{R}^n \times \mathbb{R}^n$ by $\psi_\alpha(v) = (\varphi_\alpha(\pi(v)), (\varphi_\alpha)_* v)$, where we think of $(\varphi_\alpha)_* v$, which is really a tangent vector to $\mathbb{R}^n$, as a vector in $\mathbb{R}^n$. Give TM the topology in which open sets are the unions of sets of the form $O \subset V_\alpha$ such that $\psi_\alpha(O) \subset \mathbb{R}^n \times \mathbb{R}^n$ is open. Check that ψ_α are charts, so that TM is a manifold. Check that $\pi : TM \to M$ is smooth.*

Every point v in TM is in some V_α since v is the tangent vector at some point p (as given by $p = \pi(v)$), and every point p is in some set U_α. To establish that ψ_α are charts and TM is a smooth manifold, note that φ_α is smooth and so is the pushforward, so ψ_α is smooth, too. Transition functions from one chart to another are likewise smooth, since these are compositions of smooth ψ_α and ψ_β^{-1} which overlap on some open set. The projection π is smooth because we can define the $\mathbb{R}^{2n} \to \mathbb{R}^n$ version of the map by just ignoring the last n coordinates. But this map is just $\varphi_\alpha \circ \pi \circ \psi_\alpha^{-1}$, so π must be smooth.

56 *Given bundles $\pi : E \to M$ and $\pi' : E' \to M'$, show that the maps $\psi : E \to E'$ and $\phi : M \to M'$ are a bundle morphism iff $\pi' \circ \psi = \phi \circ \pi$. Show that ψ uniquely determines ϕ.*

Bundle morphisms respect the bundle structure, i.e. the split into fiber and base. If $\pi' \circ \psi = \phi \circ \pi$, then the fiber in E over p must be mapped to the fiber in E' over $p' = \phi(p)$, or else the relationship would not hold. Conversely, if the fibers are mapped to the appropriate base points, then it doesn't matter if we do the tangent space map first or second.

57 *Check that ϕ_* is smooth when we make the tangent bundle into a manifold as in the previous exercise.*

(typo on the "previous" exercise?) A map between two manifolds is smooth if it takes smooth functions on the first manifold to smooth functions on the second. Points in the manifold in question are vectors v, so functions of them are 1-forms ω. Using an exact 1-form df we have $df(v) = v(f)$. When $v = \phi_*(u)$ we get $df(\phi_*(u)) = \phi_*(u)(f) = u(\phi^* f)) = u(f \circ \phi)$, which is smooth.

58 *Show that if $\phi : M \to M'$ is a diffeomorphism, then $\phi_* : TM \to TM'$ is a bundle isomorphism.*

When ϕ is an isomorphism, the tangent spaces T_pM and $T_\phi(p)M'$ are isomorphic, since the smooth functions on M get mapped isomorphically to smooth functions on M'. Since the pushforward is smooth and linear, it's an isomorphism of the tangent spaces.

59 *Show that for any manifold M, the tangent bundle $\pi : TM \to M$ is locally trivial.*

The induced charts ψ_α give us a trivialization, with $\mathbb{R}^n$ as the standard fiber.

60 *Describe a bundle that is not locally trivial.*

61 *Check that the tangent bundle of a manifold is a vector bundle.*

To be a vector bundle, the local trivialization must be fiberwise linear. The standard fiber for a tangent bundle is $\mathbb{R}^n$, and the trivialization is linear because it is just the pushforward (exercise 17, part I).

62 *A 1-dimensional bundle is called a (real or complex) line bundle. Check that the Möbius strip is a real line bundle if we regard the standard fiber as being $\mathbb{R}$.*

Locally the Möbius strip is just $\mathbb{R}^2$, so the trivialization is linear.

63 *Show that if a vector bundle morphism is a diffeomorphism, its inverse is a vector bundle morphism.*

64 *Show that a (smooth (typo?)) section of the tangent bundle is a vector field.*

A section of the tangent bundle assigns to each point in the base space vector in its tangent space. From this we obtain a vector field from the pointwise action of the tangent vectors. The output of the vector field is again a smooth function since the section is smooth; the directional derivative (function value) changes smoothly from point to point since the tangent vectors do.

65 *Show that $\Gamma(E)$ is a module over $C^\infty(M)$.*

The first two conditions are defined in the text. The other two are simply: $(fg)s = f(gs)$ since $(fg)s := ((fg)s)(p) = (fg)(p)s(p) = f(p)g(p)s(p) = f(gs)(p)$ for $f, g \in C^\infty(M)$.

66 *Show that every section of the Möbius strip (viewed as a real line bundle over S^1) vanishes somewhere. Conclude that hte Möbius strip hos no basis of sections, hence is not trivial.*

can't cut the möbius strip and make a cylinder unless there's a zero somewhere.

67 *Show that the dual vector bundle really is a vector bundle. Also, show that given a basis of section e_i of a vector bundle E, there is a unique dual basis e^i of sections of E^* such that for each point $p \in M$, $e^i(p)$ is the basis of E_p^* dual to the basis $e_i(p)$ of E_p.*

The basic idea, I think, is that since $E|_U$ is locally equivalent to $U \times \mathbb{R}^n$ for any open set U, then since E^* just replaces E_p with E_p^* everywhere, $E^*|_U$ must be locally equivalent to $U \times (\mathbb{R}^n)^* = U \times \mathbb{R}^n$. This makes E^* a manifold. To see that it is a vector bundle, just use the canonical dual basis when mapping E_p^* to $\mathbb{R}^n$.

68 *Show that if s is a section of a vector bundle E over M and λ is a section of E^*, there is a* *smooth function $\lambda(s)$ on M given by*

$$\lambda(s)(p) = \lambda(p)(s(p))$$

for all $p \in M$. Show that $\lambda(s)$ depends $C^\infty(M)$-linearly on λ and s.

Locally, the vector bundles are trivial, so locally the section looks like a function from M to V (V^*), where $E_p = \{p\} \times V$. We can then apply λ to s pointwise to obtain the function. It is smooth since the sections are smooth. $\lambda(s)$ is linear over λ and s since the action of the cotangent vector on the tangent vector is linear in each argument. We get the $C^\infty(M)$ part since the action is pointwise and smooth.

69 *Show that a section of the cotangent bundle is the same as a 1-form.*

A 1-form is a map taking vector fields to functions on the manifold. Interpreting vector fields as sections of the tangent bundle, means that a 1-form is a map from sections of TM to functions on M. By the previous exercise we see that a section of T^*M is just such a map, so a section of the cotangent bundle is a 1-form.

77 *Show that the conditions $g_{\alpha\alpha} = 1$ and $g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha} = 1$ imply $g_{\beta\alpha}^{-1} = g_{\alpha\beta}$. Show that for any sequences $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_m$ with $\alpha_1 = \beta_1$ and $\alpha_n = \beta_m$, they imply*

$$g_{\alpha_1\alpha_2} \cdots g_{\alpha_{n-1}\alpha_n} = g_{\beta_1\beta_2} \cdots g_{\beta_{m-1}\beta_m}.$$

Letting $\gamma = \alpha$ in the cocyle condition yields $g_{\alpha\beta}g_{\beta\alpha}g_{\alpha\alpha} = 1$. Then using $g_{\alpha\alpha} = 1$ we obtain $g_{\alpha\beta}g_{\beta\alpha} = 1$, so $g_{\beta\alpha}^{-1} = g_{\alpha\beta}$. Now observe that $g_{\alpha_1\alpha_{n-1}}g_{\alpha_{n-1}\alpha_n} = g_{\alpha_1\alpha_n}$ since multiplying by the inverse of the righthandside gives $g_{\alpha_1\alpha_{n-1}}g_{\alpha_{n-1}\alpha_n}g_{\alpha_n\alpha_1} = 1$. By induction we can expand this expression to $g_{\alpha_1\alpha_2} \cdots g_{\alpha_{n-1}\alpha_n}$. Similarly, we can start with $g_{\beta_1\beta_{m-1}}g_{\alpha_{m-1}\beta_m} = g_{\alpha_1\alpha_n}$ and obtain $g_{\beta_1\beta_2} \cdots g_{\beta_{m-1}\beta_m}$, so they must be equal.

Baez Exercises: Vector Bundle Constructions

78. Prove that if $g_{\alpha\alpha} = 1$ and $g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} = 1$ where defined, $\pi : E \to M$ is a vector bundle.

We start with the disjoint union $\bigcup_\alpha U_\alpha \times V$, where the U_α are open sets on M.

Points $(p, v) \in U_\alpha \times V$ and $(p, v') \in U_\beta \times V$ are to be identified when

$$v = \rho(g_{\alpha\beta}(p))\, v'.$$

Now we need to be able to add points in the fiber E_p above p. For a given open set U_α we have

$$(p, v_1)_\alpha + (p, v_2)_\alpha = (p, v_1 + v_2)_\alpha$$

Seen from the point of view of open set U_β the rule holds,

$$(p, g_{\beta\alpha} v_1)_\beta + (p, g_{\beta\alpha} v_2)_\beta = (p, g_{\beta\alpha} v_1 + g_{\beta\alpha} v_2)_\beta$$

$$= (p, g_{\beta\alpha}(v_1 + v_2))_\beta,$$

Since the transition function $g_{\alpha\beta}$ is linear.

Similarly, we could use an open set U_γ and mix things up:

$$(p, v_1)_\alpha + (p, g_{\beta\alpha} v_2)_\beta \overset{?}{=} (p, g_{\gamma\alpha}(v_1 + v_2))_\gamma$$

LHS: $(p, v_1)_\alpha = (p, g_{\gamma\alpha} v_1)_\gamma$

$$(p, g_{\beta\alpha} v_2)_\beta = (p, \underbrace{g_{\gamma\beta} g_{\beta\alpha}} v_2)_\gamma$$

$= g_{\gamma\alpha} g_{\alpha\gamma} g_{\gamma\beta} g_{\beta\alpha} = g_{\gamma\alpha}$ by the cocycle condition

$$(p, v_1)_\alpha + (p, g_{\beta\alpha} v_2)_\beta = (p, g_{\gamma\alpha} v_1)_\gamma + (p, g_{\gamma\alpha} v_2)_\gamma = \text{RHS}$$

which is what we wanted to prove. So the cocycle condition ensures that if we add vectors belonging to a given open set U_α, this sum can be translated into a sum of vectors corresponding to a different open set U_β in a unique way.

To define the projection operator π, just use π_α for any α:

$$\pi(p, v)_\beta = \pi_\alpha(p, g_{\alpha\beta} v)_\alpha = p$$

Locally it's trivial by construction (~~any~~ take open set U_β and we get $E \simeq U_\beta \times V$)

Finally, it's a vector bundle since under a local trivialization induced by choice of U_α, each fiber $E_p = p \times V$ can be mapped to $p \times \mathbb{R}^n$ by choosing a basis for V.

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79. Show that if $T : E_p \to E_p$ is of the form $[p, v]_\alpha \mapsto [p, d\rho(x) v]_\alpha$ for some $x \in \mathfrak{g}$
and $p \in U_\alpha \cap U_\beta$, then T is also of the form

$$[p, v']_\beta \mapsto [p, d\rho(x') v']_\beta$$

for some $x' \in \mathfrak{g}$ (typo?)

Choose $v' = g_{\beta\alpha} v$, so $[p, v]_\alpha = [p, g_{\beta\alpha} v]_\beta$

This is mapped to $[p, d\rho(x) v]_\alpha = [p, g_{\beta\alpha} d\rho(x) v]_\beta$

Picking $x' = g_{\beta\alpha} x g_{\alpha\beta}$ so $d\rho(x') = g_{\beta\alpha} d\rho(x) g_{\alpha\beta}$

Then $d\rho(x') v' = g_{\beta\alpha} d\rho(x) v$, whence the T map has the desired property.

Baez Exercises: Gauge Transformations

80. Check that if $\phi : \mathbb{R}^4 \to \mathbb{C}^3$ is a solution of

$$(\partial^\mu \partial_\mu + m^2)\phi + \lambda \phi^i \phi_i \phi = 0,$$

so is any $U_1(g)\phi$ for any $g \in SU(2)$

$\phi^i \phi_i$ is the inner product in $\mathbb{C}^3$, and as such is invariant under unitary transformations like $U_1(g)$. Further, since g doesn't depend on x^μ, $\partial_\mu U_1(g) = 0$ and therefore

$$(\partial^\mu \partial_\mu + m^2 + \lambda \phi'^i \phi'_i)\phi' = (\partial^\mu \partial_\mu + m^2 + \lambda \phi^i \phi_i)\phi'$$

$$= U_1(g)(\partial^\mu \partial_\mu + m^2 + \lambda \phi^i \phi_i)\phi = 0.$$

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81. Partition of unity...

82. Show that gauge transformations form a group.

Suppose $f, h \in \mathcal{T}$ live in G. This means $[p, v]_\alpha \xrightarrow{f} [p, g_f v]_\alpha$ for $g_f \in G$ and similarly for h and $g_h \in G$.

The product is defined by $fh(p) = f(p)h(p)$ and so we get $fh : [p, v]_\alpha \longrightarrow [p, g_{fh} v]_\alpha$ for $g_{fh} = g_f g_h \in G$. When $h = f^{-1}$ we have a trivial action, so $e = g_{fh} = g_f g_{f^{-1}}$ and thus $g_{f^{-1}} = g_f^{-1}$.

Baez Exercises: Connections

83. For $A = \sum_i T_i \otimes \omega_i$, where T_i are sections of $\mathrm{End}(E)$ and ω_i are 1-forms, show that A is well-defined, i.e. independent of how we write A as a sum $\sum_i T_i \otimes \omega_i$

Can't we just express everything in terms of a basis of sections times a basis of one-forms?

84. Prove that any connection D can be written as $D^0 + A$, for some choice of standard flat connection D^0 stemming from the choice of local trivialization, and A an $\mathrm{End}(E)$-valued 1-form.

If we choose a local trivialization, this sets D^0 as follows. From the LT we get a basis of sections e_j. Then $D^0_v(e_j) = 0$ so that

$$D^0_v s = D^0_v s^j e_j = v(s^j)\, e_j + s^j D^0_v e_j = v(s^j) e_j$$

Now let's check that $D^0 + A$ is a connection.

1. $D_v(\alpha s) = D^0_v(\alpha s) + A(v)(\alpha s) = \alpha D^0_v(s) + \alpha A(v)(s) = \alpha D_v(s)$

2. $D_v(s+t) = D^0_v(s+t) + A(v)(s+t) = D^0_v(s) + D^0_v(t) + A(v)(s) + A(v)(t)$
   $= D_v(s) + D_v(t)$

3. $D_v(fs) = D^0_v(fs) + A(v)(fs) = v(f)s + fD^0_v(s) + f A(v)(s)$
   $= v(f)s + fD_v(s)$

4. $D_{v+w}(s) = D^0_{v+w}(s) + A(v+w)(s) = D^0_v(s) + A(v)(s) + D^0_w(s) + A(w)(s)$
   $= D_v(s) + D_w(s)$

5. $D_{fv}(s) = D^\circ_{fv}(s) + A(fv)(s) = f D^\circ_v(s) + f A(v)(s)$

$= f D_v(s)$

Then we must check that if D is a connection, we can write $D = D^\circ + A$ for some $End(E)$-valued 1-form A. So, is $D - D^\circ$ an $End(E)$-valued 1-form?

1. It is linear in v, since D and D° are
2. It is linear in s:

$$D_v(fs) - D^\circ_v(fs) = \cancel{v(f)}s + f D_v s - \cancel{v(f)}s - f D^\circ_v s = f(D_v - D^\circ_v)s$$

And given D, D°, how can we define A? Choosing a basis e_j of sections and local coordinates x^μ (actually the latter specify the former) we define

$$A = A^j_{\mu i}\, e_j \otimes e^i \otimes dx^\mu \quad \text{and set} \quad A^j_{\mu i} e_j = (D - D^\circ)(\partial_\mu) e_i$$

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85. Show that $D'_v(s) = g D_v(g^{-1}s)$ is a connection

1. $D'_{\alpha v}(s) = g D_{\alpha v}(g^{-1}s) = \alpha g D_v(g^{-1}s) = \alpha D'_v(s)$
2. $D'_{v+w}(s) = g D_{v+w}(g^{-1}s) = g D_v(g^{-1}s) + g D_w(g^{-1}s) = D'_v(s) + D'_w(s)$
3. $D'_v(fs) = g D_v(g^{-1}fs) = g D_v(f g^{-1}s) = g\, v(f)\, g^{-1}s + f g D_v(g^{-1}s)$

   $= v(f) + f D'_v(s)$ $\quad$ f is a function, $g \in End(E)$
4. $D'_v(s+t) = g D_v(g^{-1}(s+t)) = g D_v(g^{-1}s + g^{-1}t) = D'_v(s) + D'_v(t)$
5. $D'_{fv}(s) = g D_{fv}(g^{-1}s) = f g D_v(g^{-1}s) = f D'_v(s)$

86 *Using a local trivialization of E over $U_\alpha \subseteq M$ write the G-connection D as the standard flat connection plus a vector potential: $D = D^0 + A$. Show that the vector potential A' for D' is given in local coordinates by*

$$A'_\mu = gA_\mu g^{-1} + g\partial\mu g^{-1}.$$

Show that since A_μ lives in $\mathfrak{g}$, so does A'_μ. (Hint: show that if A_μ lives in $\mathfrak{g}$ and $g \in G$, then $gA_\mu g^{-1}$ lives in $\mathfrak{g}$. Also show that if $g \in \mathcal{G}$, $g\partial_\mu g^{-1}$ lives in $\mathfrak{g}$.) Conclude that D' is a G-connection.

$D'_\mu(s) = gD_\mu(g^{-1}s) = gD^0_\mu(g^{-1}s) + gA_\mu(g^{-1}s)$. The local trivialization means that we have chosen a basis of sections $\{e_j\}$ such that $D^0_\mu(s^je_j) = (\partial_\mu s^j)e_j$. The action of g can be represented by $g = \rho(g)^j_k e_j \otimes e^k$ and the corresponding flat covariant derivative is just $D^0_\mu(g) = (\partial_\mu\rho(g)^j_k)e_j \otimes e^k$. Inserting these expressions into $D'_\mu(s)$ gives

$$\begin{aligned}
D'_\mu(s) &= gD_\mu(g^{-1}s) = gD^0_\mu(g^{-1}s) + gA_\mu(g^{-1}s) && (1)\\
&= g\left(\partial_\mu\rho(g^{-1})^k_j s^j\right)e_k + gA_\mu g^{-1}s = D^0_\mu(s) + g\left(\partial_\mu\rho(g^{-1})^k_j\right)s^je_k + gA_\mu g^{-1}s && (2)\\
&= D^0_\mu(s) + gD^0_\mu(g^{-1})s + gA_\mu g^{-1}s = D^0_\mu(s) + (g\partial_\mu g^{-1})s + gA_\mu g^{-1}s, && (3)
\end{aligned}$$

where in the last equality we've abuse notation somewhat. Hence the vector potential has changed according to the prescribed form. For $g \in \mathcal{G}$, $g\partial_\mu g^{-1}$ lives in $\mathfrak{g}$ for the following reason. Really this expression is $\rho(g)^j_\ell\partial_\mu\rho(g^{-1})^\ell_k(x^\nu)$ and $\rho(g)(x^\nu) = \exp\left(-c_j(x^\nu)h^j\right)$, where h^j is a generator set for the Lie algebra $\mathfrak{g}$. Thus, $g\partial_\mu g^{-1} = (\partial_\mu c_j(x^\nu))gh^jg^{-1}$, which is an element of $\mathfrak{g}$.

Baez Exercises: Holonomy

88. The covariant derivative of a vector field u along a curve $\gamma(t)$ is

$$D_{\gamma'(t)}\, u(t) = \frac{d}{dt} u(t) + A(\gamma'(t))\, u(t)$$

where $u(t)$ is the field at $\gamma(t)$ and $\frac{d}{dt}u(t)$ is really $D^0_{\gamma'(t)}\, u(t)$. Show that $D_{\gamma'(t)}\, u(t)$ defined in this manner is actually independent of the choice of local trivialization.

Since D itself is basis independent, the expression will be, too. To check that it is basis independent, we'd have to know the transformation rule for $A \to A'$. But this just comes from the invariance of D itself: $D_v = D^0_v + A(v) = D'^0_v + A'(v)$; thus A will automatically have the required transformation properties. In other words, there's no point in checking! In any case, the transformation rule is, clearly,

$$A'(v)\, s = A(v)s + (D^0_v - D'^0_v)(s)$$

In coordinates we set $[A(v)]^j_{\ k} = e^j(A(v)e_k)$, so

$$[A'(v)]^j_{\ k} = [A(v)]^j_{\ k} + e^j(D^0_v(e_k) - D'^0_v(e_k))$$

Supposing $e_k = \Lambda^\ell_{\ k} e'_\ell$, the second term is

$$\delta^j_{\ k} - e^j D'^0_v(\Lambda^\ell_{\ k} e'_\ell) = \delta^j_{\ k} - v(\Lambda^\ell_{\ k})\, e^j(e'_\ell) - \Lambda^\ell_{\ k} e^j(e'_\ell)$$

$$= \delta^j_{\ k} - v(\Lambda^\ell_{\ k})(\Lambda^{-1})^j_{\ \ell} - \Lambda^\ell_{\ k}(\Lambda^{-1})^j_{\ \ell} = -v(\Lambda^\ell_{\ k})(\Lambda^{-1})^j_{\ \ell}$$

$$\Rightarrow [A'(v)]^j_{\ k} = [A(v)]^j_{\ k} - v(\Lambda^\ell_{\ k})(\Lambda^{-1})^j_{\ \ell}$$

89. Check convergence...

90. Let $\alpha : [0,T] \to M$ be a piecewise smooth path and let $f : [0,S] \to [0,T]$ be any piecewise smooth function with $f(0) = 0$, $f(S) = T$. Let β be the reparameterized path given by $\beta(t) = \alpha(f(t))$. Show that for any connection D on a vector bundle $\pi : E \to M$, $H(\alpha, D) = H(\beta, D)$

The goal here is to show that the holonomy only depends on the path, not the curve, i.e. not on the particulars of the parameterization. This will hold if the covariant derivative is zero independent of the param. One has to be pretty careful with the notation here. For a given vector field u, let's call $u_\alpha(t)$ the vector in the fiber above $\alpha(t)$ and $u_\beta(t)$ the vector in the fiber above $\beta(t)$. Then we just check if

$$D_{\alpha'(t)} u_\alpha(t) \stackrel{?}{=} D_{\beta'(t)} u_{\beta(t)} = 0 \quad \text{for } \beta(t) = \alpha(f(t)).$$

First, $\beta'(t) = \frac{d}{dt} \alpha(f(t)) = \underbrace{f'(t)}_{\text{function}} \underbrace{\alpha'(f(t))}_{\text{vector}}$

while $u_\beta(t) = u_\alpha(f(t))$. u at $\alpha(f(t)) = \beta(t)$

$$\Rightarrow D_{\beta'(t)} u_\beta(t) = D_{\beta'(t)} u_\alpha(f(t)) = \frac{d}{dt} u_\alpha(f(t)) + A(\beta'(t))\, u_\alpha(f(t))$$
$$= \frac{d}{ds} u_\alpha(s) \Big|_{s=f(t)} f'(t) + A\big(f'(t)\,\alpha'(f(t))\big)\, u_\alpha(f(t))$$
$$= f'(t) \Big[\underbrace{\frac{d}{ds} u_\alpha(s) + A(\alpha'(s))\, u_\alpha(s)}_{=0} \Big]_{s=f(t)} = 0.$$

So we have shown that if u is parallel-transported along $\alpha(t)$, it is also parallel-transported along $\beta(t)$.

There's an even easier way to see this using the path-ordered exponential:

$$H(\gamma, D) = P e^{-\int_0^t dt\, A(\gamma'(t))}$$

Choose $\gamma'(t) = \beta'(t) = f'(t)\,\alpha'(f(t))$. Then $dt\, A(\beta'(t)) = dt\, f'(t)\, A(\alpha'(f(t)))$, which is $ds\, A(\alpha'(s))$ for $s = f(t)$. Thus

$$H(\beta, D) = P e^{-\int_0^T dt\, A(\beta'(t))} = P e^{-\int_0^S ds\, A(\alpha'(s))} = H(\alpha, D)$$

---

91. Check the identities $H(1_q \alpha, D) = H(\alpha, D)$, $H(\alpha 1_p, D) = H(\alpha, D)$, and $H(1_p, D) = 1$.

We only need to check the last one, since then the other two follow. Since $1_p(t) = p$ for all t, it follows that $1_p'(t) = 0$ since

$$1_p'(t) f = \frac{d}{dt}(f(1_p(t))) = \frac{d}{dt} f(p) = 0$$

Thus $A(1_p'(t)) = 0$ and therefore the equation for the covariant derivative becomes

$$D_{1_p'(t)}\, u(t) = \frac{d}{dt} u(t) = 0$$

so $u(t) = u(0)$ and therefore $H(1_p(t), D) = 1$.

$H(1_q \alpha, D) = H(1_q, D) H(\alpha, D) = H(\alpha, D)$ and similarly for $H(\alpha 1_p, D)$.

---

92. Check that the formula for gauge transformed holonomies,

$$H(\gamma, D') = g(\gamma(T))\, H(\gamma, D)\, g(\gamma(0))^{-1}$$

holds even when the path γ does not stay within an open set over which we have trivialized the G-bundle E, by breaking up γ into smaller paths.

Well, the holonomy doesn't depend on the local trivialization, since the covariant derivative doesn't, so we just break up the path in the different local trivializations, and the result will be independent of the choices of local triv:

$$H(\gamma, D') = H(\gamma_2, D')\, H(\gamma_1, D')$$
$$= g(\gamma_2(T))\, H(\gamma_2, D)\, g(\gamma_2(0))^{-1}\, g(\gamma_1(T))\, H(\gamma_1, D)\, g(\gamma_1(0))^{-1}$$
$$= g(\gamma_2(T))\, H(\gamma_2, D)\, H(\gamma_1, D)\, g(\gamma_1(0))^{-1} = g(\gamma(T))\, H(\gamma, D)\, g(\gamma(0))^{-1}$$

---

93. Show that if D is a G-connection on a G-bundle and γ is a loop, the holonomy $H(\gamma, D)$ lives in G.

D is a G-connection when the components A_μ of the vector potential live in $\mathfrak{g}$ (the Lie algebra of G). Then, according to the path integral formula, the holonomy is an operator generated by elements of $\mathfrak{g}$, so it must live in the Lie group G.

Chapter 3

Baez Exercises: Curvature

94. Check that $[v, fw] = f[v,w] + v(f)\,w$

Apply to a function h:

$$[v,fw]h = v\circ(fw)h - fwvh = v(f(wh)) - fwvh$$
$$= v(f)\,wh + f\,vwh - fwvh$$
$$= \big(v(f)\,w + f[v,w]\big)h \qquad \text{as intended.}$$

95. Prove that $H(\gamma, D) = 1 - \varepsilon^2 F_{\mu\nu}$ for γ an infinitesimal loop along the coordinates x^μ, x^ν

We use the path-ordered exponential:

$$H(\gamma, D) = P\exp\left[-\int_0^S A(\gamma'(s);\gamma(s))\,ds\right]$$

where by $A(\gamma'(s),\gamma(s))$ we denote the vector potential, as a function of the vector field $\gamma'(s)$ and the position $\gamma(s)$. The latter is implicitly contained in the former, but in this calculation we must be careful to be explicit.

Now consider the curve γ:

$$\gamma(t) = \begin{cases} (4t\varepsilon, 0) & 0 \le t \le 1/4 \\ (\varepsilon, \varepsilon(4t-1)) & 1/4 < t \le 1/2 \\ (\varepsilon(3-4t), \varepsilon) & 1/2 \le t \le 3/4 \\ (0, 4\varepsilon(1-t)) & 3/4 \le t \le 1 \end{cases}$$

where (a,b) means $x^\mu = a$, $x^\nu = b$. It's immediately obvious that

$$\gamma'(t) = \begin{cases} 4\varepsilon\,\partial_\mu & 0 \le t \le 1/4 \\ 4\varepsilon\,\partial_\nu & 1/4 < t \le 1/2 \\ -4\varepsilon\,\partial_\mu & 1/2 \le t \le 3/4 \\ -4\varepsilon\,\partial_\nu & 3/4 \le t \le 1 \end{cases}$$

Now we're ready to start.

$$H(\gamma, D) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} P\left(\int_0^1 ds\; A(\gamma'(s); \gamma(s))\right)^n$$

$$= 1 - \int_0^1 ds\; A(\gamma'(s);\gamma(s)) + \frac{1}{2} P\left(\int_0^1 ds\; A(\gamma'(s);\gamma(s))\right)^2 + \text{terms of higher order}$$

since $A(\gamma'(s);\gamma(s)) \propto \varepsilon$, we don't need to keep any more terms

Let's concentrate on the first integral:

$$\int_0^1 ds\; A(\gamma'(s);\gamma(s)) = \int_0^{1/4} ds\; 4\varepsilon\, A_\mu(4s\varepsilon, 0) + \int_{1/4}^{1/2} ds\; 4\varepsilon\, A_\nu(\varepsilon, \varepsilon(4s-1))$$
$$- \int_{1/2}^{3/4} ds\; 4\varepsilon\, A_\mu(\varepsilon(3-4s), \varepsilon) - \int_{3/4}^{1} ds\; 4\varepsilon\, A_\nu(0, 4\varepsilon(1-s))$$

here $A_\rho(a,b)$ is shorthand for $A(\partial_\rho; (a,b))$

Since ε is small, we can Taylor expand the integrands:

$$A_\mu(4s\varepsilon, 0) = A_\mu(0,0) + \varepsilon \frac{d}{d\varepsilon'} A_\mu(4s\varepsilon', 0)\Big|_{\varepsilon'=0} + O(\varepsilon^2)$$

$$= A_\mu(0,0) + 4\varepsilon s\, \partial_\mu A_\mu\big|_{(0,0)} + O(\varepsilon^2)$$

when there's no point specified, it's $(0,0)$

$$A_\nu(\varepsilon, \varepsilon(4s-1)) = A_\nu(0,0) + \varepsilon\,\partial_\mu A_\nu + \varepsilon(4s-1)\,\partial_\nu A_\nu + O(\varepsilon^2)$$

$$A_\mu(\varepsilon(3-4s), \varepsilon) = A_\mu + \varepsilon(3-4s)\,\partial_\mu A_\mu + \varepsilon\,\partial_\nu A_\mu + O(\varepsilon^2)$$

$$A_\nu(0, 4\varepsilon(1-s)) = A_\nu + 4\varepsilon(1-s)\,\partial_\nu A_\nu + O(\varepsilon^2)$$

$$\int_0^1 ds\, A(\gamma'(s);\gamma(s)) = \int_0^{1/4} ds\left(4\varepsilon A_\mu + 16\varepsilon^2\partial_\mu A_\mu s\right) + \int_{1/4}^{1/2} ds\left(4\varepsilon A_\nu + 4\varepsilon^2\partial_\mu A_\nu + 4\varepsilon^2(4s-1)\partial_\nu A_\nu\right)$$

$$- \int_{1/2}^{3/4} ds\left(4\varepsilon A_\mu + 4\varepsilon^2(3-4s)\partial_\mu A_\mu + 4\varepsilon^2\partial_\nu A_\mu\right) - \int_{3/4}^{1} ds\left(4\varepsilon A_\nu + 16\varepsilon^2(1-s)\partial_\nu A_\nu\right)$$

$$= \varepsilon A_\mu + \frac{1}{2}\varepsilon^2\partial_\mu A_\mu + \varepsilon A_\nu + \varepsilon^2\partial_\mu A_\nu + \frac{1}{2}\varepsilon^2\partial_\nu A_\nu$$

$$- \varepsilon A_\mu - \frac{1}{2}\varepsilon^2\partial_\mu A_\mu - \varepsilon^2\partial_\nu A_\mu - \varepsilon A_\nu - \frac{1}{2}\varepsilon^2\partial_\nu A_\nu$$

$$= \varepsilon^2\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)$$ so far, so good.

Now for the second integral:

$$\frac{1}{2}P\left(\int_0^1 ds\, A(\gamma'(s);\gamma(s))\right)^2 = \int_{1>s_1>s_2>0} ds_1\, ds_2\, A(\gamma'(s_1);\gamma(s_1))\, A(\gamma'(s_2);\gamma(s_2))$$

$$= \int_0^1\int_0^{s_1} ds_2\, ds_1\, A(s_1)\, A(s_2)$$

We can also Taylor expand the integrand here, but because we only want to keep terms up to ε^2, we'll only need the zeroth-order term. That is, we can treat A as constant over the different pieces of the path.

First look at the inside integral:

$$\int_0^{s_1} ds_2\, A(s_2) = 4\varepsilon\begin{cases} s_1 A_\mu & 0\le s_1\le 1/4 \\ \frac{1}{4}A_\mu + (s_1-\frac{1}{4})A_\nu & 1/4\le s_1\le 1/2 \\ \frac{1}{4}A_\mu + \frac{1}{4}A_\nu - (s_1-\frac{1}{2})A_\mu & 1/2\le s_1\le 3/4 \\ \frac{1}{4}A_\nu - (s_1-\frac{3}{4})A_\nu & 3/4\le s_1\le 1 \end{cases}$$

Inserting this in the s_1 integral gives

$$\int_0^1 ds_1 \int_0^{s_1} ds_2\; A(s_1)\,A(s_2) = 16\varepsilon^2 \Bigg[\int_0^{1/4} ds\; s\, A_\mu A_\mu + \int_{1/4}^{1/2} ds\; A_\nu \left(\tfrac{1}{4} A_\mu + \left(s - \tfrac{1}{4}\right) A_\nu \right)$$

$$- \int_{1/2}^{3/4} ds\; A_\mu \left(\tfrac{1}{4} A_\mu + \tfrac{1}{4} A_\nu - \left(s - \tfrac{1}{2}\right) A_\mu \right) - \int_{3/4}^{1} ds\; A_\nu \left(\tfrac{1}{4} A_\nu - \left(s - \tfrac{3}{4}\right) A_\nu \right) \Bigg]$$

$$= \cancel{\tfrac{1}{2}\varepsilon^2 A_\mu^2} + \varepsilon^2 A_\nu A_\mu + \cancel{\tfrac{1}{2}\varepsilon^2 A_\nu A_\nu} - \cancel{\varepsilon^2 A_\mu^2} - \varepsilon^2 A_\mu A_\nu + \cancel{\tfrac{1}{2}\varepsilon^2 A_\mu^2} - \cancel{\varepsilon^2 A_\nu^2} + \cancel{\tfrac{1}{2}\varepsilon^2 A_\nu^2}$$

$$= \varepsilon^2 \left[A_\nu, A_\mu \right]$$

Altogether we have

$$H(\gamma, D) = 1 - \varepsilon^2 \left(\partial_\mu A_\nu - \partial_\nu A_\mu \right) - \varepsilon^2 \left[A_\mu, A_\nu \right] = 1 - \varepsilon^2 F_{\mu\nu}$$

---

56. Show that the holonomies along any two homotopic paths are the same if the curvature vanishes.

Let's choose a homotopy γ_s between two paths γ_0 and γ_1 and show that $\frac{d}{ds} H(\gamma_s, D) = 0$ if D is flat.

$$\frac{d}{ds}\left(u - \int_0^t dt_1\, A(\gamma_s'(t_1); \gamma_s(t_1))\, u + \int_0^t\!\!\int_0^{t_1} dt_1\, dt_2\; \underbrace{A(\gamma_s'(t_1); \gamma_s(t_1))\, A(\gamma_s'(t_2); \gamma_s(t_2))}\, u + \cdots \right)$$

$$= -\int_0^t dt_1\, \frac{d}{ds} A(\gamma_s'(t_1); \gamma_s(t_1))\, u + \int_0^t\!\!\int_0^{t_1} dt_1\, dt_2\, \frac{d}{ds}\Big[\qquad \Big]\, u + \cdots$$

$$= -\int_0^t dt_1\, A'\, u + \int_0^t\!\!\int_0^{t_1} dt_1\, dt_2 \left(A'(t_1)\, A(t_2) + A(t_1)\, A'(t_2) \right) u + \cdots$$

$$H(\gamma_s, D) = \sum_{n=0}^{\infty} (-1)^n \int_{t \geq t_1 \geq \cdots \geq t_n \geq 0} A(\gamma_s'(t_1))\, A(\gamma_s'(t_2)) \cdots A(\gamma_s'(t_n))\, dt_n \cdots dt_1$$

Differentiating, we obtain

$$\frac{d}{ds} H(\gamma_s, D) = \sum_{n=0}^{\infty} (-1)^n \int_{t \geq t_1 \geq \cdots \geq t_n \geq 0} \left(A_1' A_2 \cdots A_n + A_1 A_2' \cdots A_n + \cdots + A_1 A_2 \cdots A_n' \right) dt_n \cdots dt_1$$

using the shorthand $A_k' = \frac{d}{ds} A(\gamma'(t_k))$

more to come (at least include the case of Abelian vector potential)

97. Show that every connection on a bundle $\pi : E \to M$ is flat if M is one dimensional.

The connection is flat when the curvature vanishes:

$$F(u,v) = [D_u, D_v] - D_{[u,v]} = 0$$

Since M is one dimensional, in any open cet $\mathcal{U}$ we have but one coordinate, say x, and one associated coordinate vector field, ∂_x, which is a basis for vector fields on $\mathcal{U}$ (having functions as components). Thus $u = u'(x)\partial_x$ and $v = v'(x)\partial_x$ so that

$$F(u,v) = u'(x)v'(x)\, F(\partial_x, \partial_x) = 0 \quad \text{since } F \text{ is antisymmetric.}$$

---

98. Use the results of exercise 72 to show that any E-valued differential form can be written – not necessarily uniquely – as a sum of those of the form $s \otimes \omega$, where s is a section of E and ω is an ordinary differential form on M.

Isn't the point of these "not nec. uniquely" exercises the same as the HJW theorem of QIT? In other words, for the vector space $V = U \otimes W$, we can choose a basis for V by choosing bases for U and W and multiplying them. Then any element of V can be expressed uniquely. But there's still the freedom to use an overcomplete set of product vectors in the decomposition, and all decompositions will be related by unitaries in the manner of the HJW theorem.

Here, E is a vector bundle, so its got a basis, and the space of forms certainly does, too, so the above arguments apply.

99. Using the previous exercise, show that there is a unique way to define the wedge product of an E-valued form, and an ordinary form such that the wedge of the E-valued form $s \otimes \omega$ and the ordinary form μ is given by

$$(s\otimes\omega)\wedge\mu = s\otimes(\omega\wedge\mu)$$

and such that the wedge product depends $C^\infty(M)$-linearly on each factor.

We shall work in a product basis, define the wedge product by the formula given, and then show that the resulting expression is basis-independent. Start from an E-valued form α:

$$\alpha = \sum_{jk} b_{jk} s_j \otimes \omega_k = \sum_{jk} c_{jk} s'_j \otimes \omega'_k$$

for some bases $\{s_j\}$, $\{\omega_j\}$ and $\{s'_k\}$, $\{\omega'_k\}$. The bases are related by $s_j = T_j^{\;k} s'_k$, $\omega_j = R_j^{\;k}\omega'_k$. Choosing the first, define

$$\alpha\wedge\mu = \sum_{jk} b_{jk} s_j \otimes (\omega_k\wedge\mu) \qquad \text{for } \mu \text{ an ordinary form.}$$

Now change the basis.

$$\alpha\wedge\mu = \sum_{jk} b_{jk} T_j^{\;l} s'_l \otimes (R_k^{\;m}\omega'_m\wedge\mu)$$

The wedge product is linear, so

$$\alpha\wedge\mu = \sum_{\substack{jk\\ lm}} \underbrace{b_{jk} T_j^{\;l} R_k^{\;m}}_{c_{lm}} s'_l \otimes (\omega'_m \wedge\mu)$$

$$= \sum_{lm} c_{lm} s'_l \otimes (\omega'_m\wedge\mu)$$

which is what we would have obtained by defining the product in the other basis. Thus the product is independent of basis, and $C^\infty(M)$-linear by construction.

100. Show that the two definitions of the exterior covariant derivative are equivalent.

On the one hand we define d_D by saying $(d_D s)(v) = D_v s$ for any v. On the other, if we work in a coordinate system x^μ, then we have $d_D s = D_\mu s \otimes dx^\mu$. To see that these are equivalent, act on an arbitrary vector:

$$(d_D s)\, v = (D_\mu s \otimes dx^\mu)\, v = v(x^\mu) D_\mu s = v^\nu \partial_\nu (x^\mu) D_\mu s = v^\mu D_\mu s$$
$$= D_v s \qquad \text{as intended.}$$

---

101. Check that the (wedge) product of End(E)-valued forms with E-valued forms is well-defined by its action on products:

$$(T \otimes \omega) \wedge (s \otimes \mu) = T(s) \otimes (\omega \wedge \mu)$$

See 98 and 99. Everything is linear, so the choice of basis doesn't matter.